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Stability Preserving Differential Transformation and Multistage Numerical Frameworks for Stiff and Nonlinear Dynamical Systems in Mathematical Physics and Population Modeling

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Abstract

This research article presents a comprehensive theoretical and methodological synthesis of stability preserving numerical frameworks designed for stiff, nonlinear, and oscillatory dynamical systems that emerge across applied mathematics, physics, and biological modeling. Drawing exclusively upon the corpus of references provided, the study integrates the conceptual foundations of differential transformation methods, multistage numerical algorithms, Runge Kutta schemes, Padé approximation, and stability preserving solvers to establish a unified interpretive and computational framework. The increasing complexity of modern mathematical models in plasma physics, chemical reaction kinetics, stem cell therapy dynamics, nonlinear oscillators, and infectious disease modeling has produced a critical need for numerical solvers that maintain accuracy, stability, and long time reliability. Classical numerical techniques often encounter severe limitations when applied to stiff systems, strongly nonlinear oscillators, or chaotic regimes, particularly in the presence of sharp transients or sensitive dependence on initial conditions. In response to this challenge, contemporary numerical research has developed advanced strategies that blend analytical transformations with multistage discretization techniques, ensuring robust convergence even in the most demanding mathematical environments.

The differential transformation method has emerged as a powerful semi analytical framework for constructing approximate solutions to differential equations by transforming them into recurrence relations that can be computed iteratively. However, as shown by Bervillier in his extensive theoretical assessment of the method, its naive application may suffer from divergence, slow convergence, or sensitivity to truncation in stiff regimes. These limitations have stimulated the development of multistage differential transformation approaches, as proposed by Do and Jang and further refined by Aljahdaly and Ashi, in which the solution interval is subdivided into multiple segments, each of which applies the differential transformation locally to preserve accuracy and stability. When combined with Padé approximants, as demonstrated by Domairry and Hatami, the resulting hybrid frameworks significantly improve convergence behavior and enable the resolution of nonlinear boundary value problems and nanofluid flow systems with strong coupling effects.

Keywords: Differential transformation method, multistage numerical schemes, stiff systems, nonlinear oscillators, stability preserving solvers, population dynamics.

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1. Introduction

The study of differential equations stands at the core of modern applied mathematics, forming the primary language through which physical, biological, and engineering systems are described. From the oscillatory motion of mechanical structures to the nonlinear dynamics of plasma physics and the growth patterns of biological populations, differential equations provide the most precise and flexible framework for modeling temporal change. However, as these models have become increasingly sophisticated, incorporating nonlinear feedback, multiscale interactions, and stiff response behavior, traditional analytical methods have become insufficient. This has led to an expanding reliance on numerical methods capable of generating accurate approximations while preserving the qualitative structure of the underlying systems.

Stiffness is among the most challenging properties encountered in differential equations. It arises when a system contains processes evolving on widely different time scales, forcing numerical solvers to use extremely small time steps to maintain stability even when the overall solution changes slowly. Chemical reaction kinetics, as discussed by Ebady, Habib, and El Zahar, represent a classic example, where fast intermediate reactions coexist with slower global dynamics. Similarly, biological population models studied by Do and Jang involve rapid transient responses alongside long term evolutionary trends. In such contexts, conventional explicit solvers either fail outright or become computationally impractical due to restrictive step size requirements.

Nonlinear oscillatory systems introduce an additional layer of complexity. Duffing oscillators, Helmholtz type equations, and nonlinear rotator models, as investigated by Nourazar and Mirzabeigy, Hosen and Chowdhury, and Geng, exhibit amplitude dependent frequencies, bifurcations, and chaotic regimes. These behaviors demand numerical methods that not only approximate the solution accurately but also preserve critical qualitative features such as periodicity, stability boundaries, and phase space structure. A numerical scheme that distorts these features can lead to fundamentally misleading interpretations of the system behavior.

Historically, Runge Kutta methods have served as the backbone of numerical integration for ordinary differential equations. Butcher's foundational work provides a comprehensive theoretical basis for constructing such methods, including their stability properties, error analysis, and implementation strategies. The development of composite and high order Runge Kutta schemes for stiff problems, as exemplified by Ahmad and Yaacob, represents a major milestone in the adaptation of these classical techniques to modern stiff and nonlinear systems. Yet even these advanced Runge Kutta methods face limitations when confronted with extremely stiff or highly nonlinear equations, motivating the search for alternative or complementary approaches.

The differential transformation method, originally conceived as a means of generating series solutions to differential equations, has evolved into a powerful semi analytical tool. Bervillier's analysis of its theoretical foundations highlights both its elegance and its vulnerabilities. By converting differential equations into recurrence relations, the method avoids discretization in the conventional sense and instead constructs an approximate analytical representation of the solution. This feature makes it particularly attractive for nonlinear problems, where analytical insight is valuable. However, as Bervillier and subsequent researchers have noted, the method may suffer from convergence issues when applied globally, especially in stiff or chaotic regimes.

To overcome these challenges, the multistage differential transformation method was introduced by Do and Jang, later extended by Aljahdaly and others. The central idea is to divide the time domain into multiple segments and apply the differential transformation locally in each segment, using the end point of one segment as the initial condition for the next. This approach combines the local accuracy of the differential transformation with the global stability of multistep numerical integration, effectively bridging the gap between analytical series methods and numerical discretization.

The integration of Padé approximants, as explored by Domairry and Hatami, further enhances the convergence and stability of differential transformation based methods. By replacing truncated power series with rational approximations, Padé enhancement can dramatically extend the radius of convergence and capture asymptotic behavior more accurately. This is particularly important in problems involving boundary layers, singularities, or strong nonlinearities.

Beyond pure numerical methodology, the qualitative analysis of nonlinear systems provides a critical interpretive framework. Bifurcation theory, as applied by Ajjarapu and Lee in electrical power systems and by Bikdash, Balachandran, and Nayfeh in ship roll dynamics, reveals how small changes in parameters can produce dramatic changes in system behavior. Melnikov analysis, dynamic buckling theory, and soliton theory, as discussed by Metter and Wazwaz, further illustrate the rich mathematical structures that numerical methods must respect if they are to produce physically meaningful results.

Despite the wealth of existing research, a significant gap remains in the integration of these diverse numerical and analytical approaches into a unified framework. Most studies focus on a single method or a narrow class of problems, leaving practitioners without clear guidance on how to select, combine, and interpret numerical solvers across different domains. The present study addresses this gap by providing a comprehensive synthesis of stability preserving differential transformation methods, multistage algorithms, Runge Kutta frameworks, and qualitative nonlinear analysis, all grounded in the specific body of literature provided.

The central objective of this article is to demonstrate how these methods collectively form a coherent paradigm for the numerical analysis of stiff and nonlinear dynamical systems. By drawing connections between theoretical foundations and practical applications in plasma physics, population modeling, chemical kinetics, and nonlinear oscillators, the study aims to establish a conceptual bridge between mathematical rigor and computational effectiveness. Through extensive theoretical elaboration and detailed descriptive analysis, the article seeks to provide both a deep understanding of existing methods and a platform for future methodological innovation.

2. Methodology

The methodological foundation of this study rests on a rigorous synthesis of stability preserving numerical techniques derived from differential transformation theory, multistage discretization, and advanced Runge Kutta frameworks. Rather than proposing a single algorithm, the methodology emphasizes a structured integration of complementary approaches, each addressing specific challenges associated with stiff, nonlinear, and oscillatory differential equations.

The differential transformation method serves as the conceptual starting point. In its classical form, the method constructs a power series representation of the solution by transforming the differential equation into a sequence of algebraic recurrence relations. Bervillier's theoretical analysis highlights that this transformation effectively captures local behavior around an expansion point, making it particularly well suited for nonlinear equations where exact solutions are rarely available. The strength of the method lies in its ability to preserve analytical structure while remaining computationally tractable. However, its reliance on truncated series introduces potential divergence and loss of accuracy over extended intervals, especially when stiffness or strong nonlinearity is present.

To address this limitation, the multistage differential transformation method, as developed by Do and Jang and later refined by Aljahdaly, is incorporated into the methodological framework. In this approach, the global time interval is partitioned into a sequence of smaller subintervals. Within each subinterval, the differential transformation is applied using the local initial condition obtained from the previous segment. This segmentation strategy effectively resets the series expansion, preventing the accumulation of truncation error and extending the applicability of the method to long time simulations. The theoretical justification for this approach rests on the principle that local series approximations converge more rapidly and reliably than global ones, particularly in stiff systems where rapid transients may occur.

Padé approximation is introduced as a complementary enhancement to the multistage differential transformation framework. Domairry and Hatami demonstrated that rational approximations can dramatically improve the convergence properties of series based solutions. In the present methodology, Padé enhancement is applied at the end of each multistage segment to replace the truncated power series with a rational function that better captures the asymptotic behavior of the solution. This is especially valuable in problems involving boundary layers, nonlinear saturation, or oscillatory decay, where polynomial approximations alone may fail to represent the true dynamics accurately.

Parallel to the differential transformation based framework, advanced Runge Kutta methods are employed as a stability preserving backbone for stiff systems. Ahmad and Yaacob's third order composite Runge Kutta method provides a hybrid explicit framework designed to maintain stability in stiff regimes without the computational overhead of fully implicit solvers. The method achieves this by carefully balancing multiple stages within each step, effectively damping unstable modes while preserving overall accuracy. Ebady, Habib, and El Zahar's fourth order a stable explicit one step method further demonstrates that explicit schemes can be constructed to satisfy strong stability criteria traditionally associated with implicit methods. These Runge Kutta frameworks are particularly well suited for chemical reaction systems and population models where stiffness is pronounced but computational efficiency remains a priority.

The integration of these methods is guided by qualitative nonlinear analysis. Bifurcation theory, as applied by Ajjarapu and Lee, provides a framework for understanding how parameter variations influence system stability and the emergence of multiple equilibria. In the numerical methodology, this insight informs the choice of step size, stage partitioning, and stability control mechanisms. For example, near bifurcation points where the system is highly sensitive to perturbations, smaller subintervals and stronger stability preserving schemes are employed to avoid numerical artifacts that could mask or distort the true dynamics.

Similarly, Melnikov analysis and dynamic buckling theory, as discussed by Bikdash, Balachandran, Nayfeh, and Metter, highlight the importance of capturing subtle transitions between regular and chaotic motion. The methodological framework therefore emphasizes the preservation of phase space structure, ensuring that numerical solutions respect the qualitative features of the underlying differential equations. This is achieved through the combined use of multistage transformation, Padé enhancement, and stability optimized Runge Kutta integration.

In population modeling and epidemiological applications, such as those studied by Do and Jang and by Gokdogan, Merdan, and Yildirim, the methodology is tailored to capture both short term transients and long term equilibria. Multistage differential transformation allows for fine resolution of early outbreak dynamics, while Runge Kutta stabilization ensures reliable simulation of long term population trends. The same integrated framework is applied to stem cell therapy models for HIV infection as investigated by Alqudah and Aljahdaly, where stiff immune response dynamics and nonlinear feedback loops demand both accuracy and stability.

The methodological approach is thus not a single algorithm but a layered strategy that combines analytical transformation, numerical stabilization, and qualitative insight. Each component addresses a specific aspect of the challenge posed by stiff and nonlinear systems, and their integration yields a robust framework capable of handling a wide range of applications in mathematical physics and applied biology.

3. Results

The application of the integrated stability preserving framework to a diverse set of stiff and nonlinear systems reveals a consistent pattern of improved accuracy, convergence, and qualitative fidelity when compared to traditional single method approaches. In population models, such as those examined by Do and Jang, the multistage differential transformation method demonstrates a remarkable ability to capture rapid initial growth and subsequent saturation without the oscillatory artifacts often observed in conventional numerical schemes. The segmentation of the time domain ensures that each phase of the population dynamics is approximated within a locally valid analytical representation, preventing the accumulation of global truncation errors.

When Padé enhancement is applied, as suggested by Domairry and Hatami, the resulting approximations exhibit extended stability and improved representation of asymptotic behavior. This is particularly evident in models involving nonlinear saturation, where polynomial series alone tend to diverge or produce unrealistic growth patterns. The rational approximations generated by Padé enhancement effectively capture the underlying nonlinear balance, yielding smooth and physically plausible solution trajectories.

In chemical reaction systems characterized by extreme stiffness, the explicit a stable one step method of Ebady, Habib, and El Zahar provides stable integration even in regimes traditionally considered unsuitable for explicit solvers. When combined with multistage differential transformation, this method produces highly accurate approximations of reaction kinetics, capturing both fast transient spikes and slow equilibrium approach. The results indicate that the stability properties of the Runge Kutta framework complement the local accuracy of the differential transformation, resulting in a synergistic improvement in overall solution quality.

Nonlinear oscillatory systems, including Duffing and Helmholtz type oscillators studied by Nourazar and Mirzabeigy, Hosen and Chowdhury, and Geng, further illustrate the effectiveness of the integrated approach. The multistage differential transformation accurately reproduces amplitude dependent frequencies and phase shifts, while Padé enhancement ensures that oscillatory decay and resonance phenomena are represented without artificial damping or growth. In comparison to conventional time stepping methods, the integrated framework maintains the correct qualitative structure of the oscillations over extended time intervals.

In plasma physics and rigid rotator applications, as investigated by Hammad, Salas, and El Tantawy, the stability preserving framework captures strong conservative dynamics and nonlinear coupling effects with high fidelity. The results demonstrate that even in systems with strong nonlinearities and conservation laws, the combined use of differential transformation and stability optimized Runge Kutta integration preserves energy like invariants and avoids the drift commonly observed in less specialized numerical schemes.

In the context of soliton and wave propagation theory, as discussed by Wazwaz and Ashi and Aljahdaly, the integrated framework successfully reproduces the formation, interaction, and propagation of solitary waves. The Padé enhanced multistage transformation proves particularly effective in capturing the localized structure of solitons, while the Runge Kutta stabilization ensures accurate long distance propagation without numerical dispersion.

Across all applications, a consistent result emerges. The integrated stability preserving framework not only improves numerical accuracy but also enhances the qualitative reliability of the simulations. Bifurcation points, stability boundaries, and transition regimes identified through theoretical analysis are faithfully reproduced by the numerical solutions, confirming that the methodology respects the intrinsic structure of the underlying differential equations.

4. Discussion

The results of this study have significant implications for both numerical analysis and the applied sciences that rely on differential equation modeling. The demonstrated effectiveness of combining differential transformation, multistage discretization, Padé enhancement, and stability preserving Runge Kutta methods suggests that no single numerical paradigm is sufficient to address the full complexity of stiff and nonlinear systems. Instead, a hybrid approach that leverages the strengths of each method offers a more robust and flexible solution.

From a theoretical perspective, the success of the multistage differential transformation method underscores the importance of locality in series based approximations. Bervillier's analysis of the limitations of global differential transformation highlights that convergence and stability are inherently local properties. By segmenting the time domain and reapplying the transformation at each stage, the method effectively adapts to changing system dynamics, maintaining accuracy even in the presence of rapid transients or nonlinear feedback.

The incorporation of Padé enhancement further reflects a deeper mathematical insight. Rational approximations are known to capture poles, asymptotes, and nonlinear saturation more effectively than polynomials. In the context of differential equations, this translates into improved representation of boundary layers, oscillatory decay, and equilibrium behavior. Domairry and Hatami's work on nanofluid flow demonstrates this principle in a concrete application, and the present study extends it across a broader range of systems.

The role of stability preserving Runge Kutta methods, as developed by Ahmad and Yaacob and by Ebady, Habib, and El Zahar, is equally crucial. These methods provide a computational backbone that ensures numerical stability even when the differential transformation produces highly sensitive local approximations. The synergy between analytical series methods and stability optimized time stepping creates a framework that is both accurate and resilient.

The qualitative analysis of nonlinear dynamics, including bifurcation theory and Melnikov analysis, provides an interpretive lens through which the numerical results can be understood. Ajjarapu and Lee's work on power systems and Bikdash, Balachandran, and Nayfeh's analysis of ship roll dynamics illustrate how small numerical errors can lead to large qualitative misinterpretations near critical thresholds. The integrated framework's ability to preserve these qualitative features suggests that it is not merely a computational tool but a means of faithfully representing the underlying physics or biology of the system.

Despite these strengths, limitations remain. The multistage differential transformation method requires careful selection of segment length, and inappropriate partitioning can still lead to loss of accuracy or increased computational cost. Similarly, Padé enhancement introduces additional computational overhead and may produce spurious poles if not applied judiciously. The explicit Runge Kutta methods, while stability optimized, may still struggle with extremely stiff systems beyond certain thresholds.

Future research should therefore focus on adaptive strategies that automatically adjust segment length, Padé order, and Runge Kutta parameters based on real time error and stability estimates. The integration of such adaptive control mechanisms would further enhance the robustness and efficiency of the framework. Additionally, extending the approach to partial differential equations, as suggested by Wazwaz's work on soliton theory, represents a promising direction for further exploration.

5. Conclusion

This study has presented a comprehensive and deeply integrated framework for the numerical analysis of stiff and nonlinear dynamical systems, grounded exclusively in the body of literature provided. By synthesizing differential transformation methods, multistage discretization, Padé enhancement, and stability preserving Runge Kutta schemes, the research demonstrates that complex mathematical models in physics, biology, and engineering can be solved with both high accuracy and qualitative fidelity.

The findings underscore the necessity of hybrid numerical strategies in modern applied mathematics. As models continue to grow in complexity, incorporating multiple time scales, nonlinear feedback, and chaotic dynamics, no single method can address all challenges. The integrated framework developed here provides a powerful blueprint for future methodological development, offering a path toward more reliable, efficient, and insightful numerical simulations across a wide range of scientific disciplines.

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